DATE

UNIT :- III ASSIGNMENT

(1) Test the convergence of series.

(i) $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$

(ii) $\Sigma\left(\sqrt[3]{n^3+1} - n\right)$

Sol (i) $\Sigma u_n = \frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$

1, 3, 5 ---
n^{th} term $= 1 + (n-1)2$
$= 2n-1$
1, 2, 3 ---
n^{th} term $= n$

$$U_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$U_n = \frac{n\left(2-1/n\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$V_n = \frac{n}{n^3}$$

$V_n = \frac{1}{n^2}$ [n^{th} term of Aux. series ΣV_n]

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\cancel{n}(2-1/n)\cdot \cancel{n^2}}{\cancel{n^3}(1+1/n)(1+2/n)}$$

$\lim_{n\to\infty} \frac{2}{1\cdot1} = \frac{2}{1} = 2$ [fixed, finite, non-zero].

Comparison test is applicable.

By p-test,

$\Sigma V_n = \Sigma \frac{1}{n^2}$ is convergent

as $p = 2 > 1$

$\therefore \Sigma u_n$ is convergent.

(ii) $\Sigma\left(\sqrt[3]{n^3+1}-n\right)$

$$u_n = (n^3+1)^{1/3}-n$$
$$= \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{1/3}-n$$
$$= n\left[\left(1+\frac{1}{n^3}\right)^{1/3}-1\right]$$
$$= n\left[1+\frac{1}{3n^3}+\frac{1/3\,(1/3-1)}{2!}\left(\frac{1}{n^3}\right)^2+\cdots-1\right]$$
$$= \frac{n}{n^3}\left[\frac{1}{3}+\frac{1/3\,(1/3-1)}{2!}\cdot\frac{1}{n^3}\cdots\right]$$
$$= \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\cdots\right]$$

Let $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{V_n} = \lim_{n\to\infty}\frac{1/n^2\left[1/3-1/9n^3\cdots\right]}{1/n^2} = \frac{1}{3}$$

[finite, fixed, non-zero].

Comparison Test is applicable.

By p-test, $\Sigma V_n = \Sigma\frac{1}{n^2}$ is convergent.

as $p=2>1$

$\therefore \Sigma u_n$ is also convergent.

(2) (i) Test the series: $\frac{1}{1\cdot2\cdot3}+\frac{x}{4\cdot5\cdot6}+\frac{x^2}{7\cdot8\cdot9}+\cdots$ by using Ratio Test.

(ii) Test the series: $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7}+\cdots$

Sol^n (i) $\Sigma u_n = \frac{1}{1\cdot2\cdot3}+\frac{x}{4\cdot5\cdot6}+\frac{x^2}{7\cdot8\cdot9}+\cdots$

$$u_n = \frac{x^{n-1}}{(3n-2)(3n-1)3n}.$$

$$u_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{x^n}(3n+1)(3n+2)(3n+3)}{x(3n-2)(3n-1)(3n)\cdot\cancel{x^n}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(3n+1)(3n+2)(3n+3)}{(3n-2)(3n-1)(3n)\cdot x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\cancel{n^3}(3+1/n)(3+2/n)(3+3/n)}{\cancel{n^3}(3-2/n)(3-1/n)\,3\cdot x}$$

$$\lim_{n\to\infty}\frac{\cancel{(3)}\cancel{(3)}\cancel{(3)}}{\cancel{(3)}\cancel{(3)}\cancel{(3)}\cdot x} = \frac{1}{x}$$

Hence, by ratio test, Σu_n is convergent if $\frac{1}{x} > 1$, i.e. $x < 1$ (convergent)

if $\frac{1}{x} < 1$, i.e. $x > 1$ (divergent)

and the test fails if $\frac{1}{x} = 1$ i.e $x = 1$.

when $x = 1$, $u_n = \frac{1}{(3n-2)(3n-1)(3n)}$

$$u_n = \frac{1}{n^3(3-2/n)(3-1/n)(3)}$$

$$V_n = \frac{1}{n^3}$$

where V_n is n^{th} term of auxiliary series ΣV_n

$$\frac{u_n}{V_n} = \frac{n^3}{n^3(3-2/n)(3-1/n)(3)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(3-2/n)(3-1/n)(3)} = \frac{1}{27}$$

which is fixed, finite and non-zero quantity.
Hence, Comparison test can be applied.
But $\sum \frac{1}{n^3}$ is convergent as $p = 3 > 1$
(by p-test)
Hence, the given series $\sum u_n$ is also convergent.
Finally, $\sum u_n$ is convergent if $x \leq 1$ and divergent if $x > 1$.

(ii) $\sum u_n = \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots$

$$u_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)\, x^{2n-1}}{2\cdot 4\cdot 6 \cdots 2n(2n-1)}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)\, x^{2n+1}}{2\cdot 4\cdot 6 \cdots 2n(2n+2)(2n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{1\cdot 3\cdot 5 \cdots (2n-1)}\, x^{2n} \,(\cancel{2\cdot 4\cdot 6 \cdots 2n})(2n+2)(\cancel{2n+1})}{x(\cancel{2\cdot 4\cdot 6 \cdots 2n})(2n-1)(\cancel{1\cdot 3\cdot 5 \cdots 2n-1})(\cancel{2n+1})\, x^{2n}\cdot x}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{x^2(2n-1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{n(2+2/n)}{x^2\cdot n(2-1/n)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2+2/n)}{x^2(2-1/n)} = \frac{\cancel{2}}{\cancel{2}x^2} = \frac{1}{x^2}$$

Hence, by ratio test, $\sum u_n$ is
Convergent if $\frac{1}{x^2} > 1$, i.e. $x^2 < 1$

divergent if $\frac{1}{x^2} < 1$, i.e. $x^2 > 1$

and the test fails if $\frac{1}{x^2} = 1$, i.e. $x^2 = 1$.

When $x^2 = 1 \Rightarrow x = 1$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)}{(2n-1)}$$

$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = n\left[\frac{2n+2}{2n-1} - 1\right]$$

$$= n\left[\frac{\cancel{2n}+2-\cancel{2n}+1}{2n-1}\right]$$

$$= \frac{3n}{2n-1}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right] = \frac{\cancel{n}\,3}{\cancel{n}(2(-1/n)} = \frac{3}{2}$$

[finite, fixed, non-zero].

Hence, the given series is convergent at $x = 1$.
Hence, the given series ΣU_n is also convergent
Finally, ΣU_n is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

(3) Find the fourier series to represent the function $f(x)$ given by: $f(x) = \begin{cases} -k, & \text{for } -\pi < x < 0 \\ k, & \text{for } 0 < x < \pi \end{cases}$

Hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — (1)

then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$a_0 = \frac{1}{\pi}\left[-k[x]_{-\pi}^{0} + k[x]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-k[0+\pi] + k[\pi-0]\right]$$

$$= \frac{1}{\pi}\left[-k\pi + k\pi\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + k\left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-k[0+0] + k[0-0]\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + k\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[k\left[\frac{\cos 0}{n} - \frac{\cos n(-\pi)}{n}\right] - k\left[\frac{\cos n\pi}{n} - \frac{\cos 0}{n}\right]\right]$$

$$= \frac{1}{\pi}\left[k\left[\frac{1}{n} - \frac{(-1)^n}{n}\right] - k\left[\frac{(-1)^n}{n} - \frac{1}{n}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{k}{n} - \frac{k(-1)^n}{n} - \frac{k(-1)^n}{n} + \frac{k}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2k}{n} - \frac{2k(-1)^n}{n}\right]$$

$$\boxed{b_n = \frac{1}{\pi}\left[\frac{2k}{n}\left(1-(-1)^n\right)\right]}$$

Sol:

Now from eq^n (1)

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\pi}\frac{2k}{n}\left[1-(-1)^n\right] \sin nx$$

$$f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1-(-1)^n\right] \sin nx$$

$$f(x) = \frac{2k}{\pi}\left[2\sin x + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5} + \cdots\right]$$

$$f(x) = \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

Now, Put $x = \frac{\pi}{2}$, $0 < \frac{\pi}{2} < \pi$, $f(x) = k$

$$k = \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

$$\frac{\pi}{4} = \sin\frac{\pi}{2} + \sin\frac{3\pi}{2}\cdot\frac{1}{3} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots$$

$$\frac{\pi}{4} = 1 + \frac{(-1)}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Proved

(1) Obtain the fourier series for the function:

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$$, and hence

Show that, a.) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}$

DATE

b) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots \frac{\pi^2}{8}$.

Sol: a) Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$ — (1)

then, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} x\,dx+\int_{0}^{\pi} -x\,dx\right]$$

$$=\frac{1}{\pi}\left[\left[\frac{x^2}{2}\right]_{-\pi}^{0}-\left[\frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[0-\frac{\pi^2}{2}-\frac{\pi^2}{2}+0\right]$$

$$=\frac{1}{\cancel{\pi}}\left[-\frac{\cancel{2}\pi^{\cancel{2}}}{\cancel{2}}\right]$$

$$\boxed{a_0=-\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\,dx+\int_{0}^{\pi} -x\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right]_{-\pi}^{0}-\left[x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right]_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\left[0+\frac{1}{n^2}-0-\frac{(-1)^n}{n^2}\right]-\left[0+\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]\right]$$

$$=\frac{1}{\pi}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}-\frac{(-1)^n}{n^2}+\frac{1}{n^2}\right]$$

$$=\frac{1}{\pi}\left[\frac{2}{n^2}-\frac{2(-1)^n}{n^2}\right]$$

$$\boxed{a_n=\frac{2}{n^2\pi}\left[1-(-1)^n\right]}=\begin{cases}0, & n \text{ is even}\\ \frac{4}{n^2\pi}, & n \text{ is odd}.\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin nx\,dx + \int_{0}^{\pi} -x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n}\right]_{-\pi}^{0} - \left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + 0 - \left[(-\pi)\left(-\frac{\cos n\pi}{n}\right) - 0\right] - \pi\left(-\frac{\cos n\pi}{n}\right) + 0\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n} + \pi\frac{\cos n\pi}{n}\right]$$

$$\boxed{b_n = 0}$$

from eq^n (1)

$$f(x) = \frac{-\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\cos nx + 0$$

$$f(x) = \frac{-\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] \text{ — (2)}$$

(a) $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \text{ — (3)}$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{2^2}{(2n)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$\frac{\pi^2}{8} = 4\sum_{n=1}^{\infty} \frac{1}{(2n)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$\frac{\pi^2}{8} = 3\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

DATE

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{24} \quad \text{--- (4)}$$

On (3) - (4)

$$\frac{\pi^2}{8} - \frac{\pi^2}{24} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$\frac{3\pi^2 - \pi^2}{24} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

$$\boxed{\frac{\pi^2}{12} = \frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots}$$

b.) At the point of discontinuity,

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right]$$

$$= \frac{1}{2}(0-0)$$

$$= 0$$

Putting $x=0$ in eq (2)

$$f(0) = \frac{-\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$0 = \frac{-\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots}$$

(5) Find the half range cosine series of the function:

$$f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2 \end{cases}$$

Sol- Let Half range cosine series is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi t}{l}, \quad \text{Here } l = 2.$$

$$a_0 = \frac{2}{l}\int_0^{l} f(t)\,dt$$

$$= \frac{2}{2}\left[\int_0^1 2t\,dt + \int_1^2 2(2-t)\,dt\right]$$

$$= 2\left[\frac{t^2}{2}\right]_0^1 + 2\left[2t - \frac{t^2}{2}\right]_1^2$$

$$= 1 + 2\left[4 - 2 - \left[2 - \frac{1}{2}\right]\right]$$

$$= 1 + 2\left[\cancel{2} - \cancel{2} + \frac{1}{2}\right]$$

$$= 1 + 1$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{l}\int_0^2 f(x)\,\cos\frac{n\pi t}{l}\,dt$$

$$= \frac{2}{2}\left[\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt\right]$$

$$= 2\left[t\,\frac{\sin n\pi t}{2} \times \frac{2}{n\pi} - \frac{2}{n\pi}\left(-\frac{\cos n\pi t}{2}\right) \times \frac{2}{n\pi}\right]_0^1$$

$$+ 2\left[(2-t)\,\frac{\sin n\pi t}{2} \times \frac{2}{n\pi} + \frac{2}{n\pi} \times \left(-\frac{\cos n\pi t}{2}\right) \times \frac{2}{n\pi}\right]_1^2$$

$$= 2\left[\frac{\sin n\pi}{2} \times \frac{2}{n\pi} + \frac{4}{(n\pi)^2}\,\frac{\cos n\pi}{2} + \frac{4}{(\pi n)^2}\,(-\cos 0)\right]$$

$$+ \left[\frac{4}{(n\pi)^2}\left(-\frac{\cos n\pi(2)}{2}\right) - \left[\frac{\sin n\pi}{2} \times \frac{2}{n\pi} - \frac{4}{(n\pi)^2}\,\frac{\cos n\pi}{2}\right]\right]$$

$$= 2\left[\frac{4}{(n\pi)^2}\cos\frac{n\pi}{2} - \frac{4}{(n\pi)^2}\right] + 2\left[\frac{4}{(n\pi)^2}\left(-\frac{\cos 2n\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\frac{\pi n}{2}\right]$$

$$= \frac{8}{(n\pi)^2}\cos\frac{n\pi}{2} - \frac{8}{(n\pi)^2} - \frac{8}{(\pi n)^2}\cos\frac{n\pi \cancel{2}}{\cancel{2}} + \frac{8}{(n\pi)^2}\cos\frac{n\pi}{2}$$

$$= \frac{16}{(n\pi)^2}\cos\frac{n\pi}{2} - \frac{8}{(n\pi)^2}(-1)^n - \frac{8}{(n\pi)^2}$$

$$a_n = \frac{8}{(n\pi)^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$\boxed{f(t) = 1 + \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]\cos\frac{n\pi t}{2}}$$